数理解析研究所講究録
第 1558 巻 2007 年 43-64

On the local well-posedness of the Cauchy problem for the Schrödinger map

名古屋大学大学院 多元数理科学研究科 加藤 淳 (Jun Kato)
Graduate School of Mathematics, Nagoya University

1 Introduction

In this note, we consider the local well-posedness of the Cauchy problem for the Schrödinger map for the low regularity initial data. In particular, we give the refined version of the existence theorem compared with the one derived in [8]. We also give the outline of the proof of the uniqueness result in [9].

1.1 Schrödinger map

The Schrödinger map from $\mathbf{R}\times\mathbf{R}^n$ to S^2 is formulated as follows. To begin with, we identify S^2 with the complex plane $\mathbf{C}$ with the specific metric by using the stereographic projection as follows. The stereographic projection $\phi: S^2\setminus\{(0,0,1)\}\to\mathbf{C}$ maps $w\in\mathbf{C}$ to

$$\phi^{-1}(w)=\Big(\frac{2\operatorname{Re}w}{1+|w|^2},\ \frac{2\operatorname{Im}w}{1+|w|^2},\ \frac{1-|w|^2}{1+|w|^2}\Big)\in S^2.$$

Here, for a complex number w, $\operatorname{Re}w$ and $\operatorname{Im}w$ denotes the real part of w and the imaginary part, respectively. Using this relation, we identify S^2 with $(\mathbf{C}, g\,dw\,d\overline{w})$ where g is given by $g(w,\overline{w})=2/(1+|w|^2)^2$.

The energy of the map $z:\mathbf{R}^n\to(\mathbf{C}, g\,dw\,d\overline{w})(\simeq S^2)$ is given by

$$E[z]=\int_{R^n}\frac{|\nabla z(x)|^2}{(1+|z(x)|^2)^2}\,dx. \tag{1.1}$$

Then, the Euler-Lagrange equation of the energy functional $E[z]$ is determined by

$$\sum_{j=1}^{n}\nabla_j\partial_j z=0, \tag{1.2}$$

where

$$\nabla_j = \partial_j - \frac{2}{1+|z|^2}\,\overline{z}\,\partial_j z. \tag{1.3}$$

The map satisfying the equation (1.2) is known as the harmonic map, and the map satisfying its evolution of the form

$$\partial_t z = i\sum_{j=1}^{n}\nabla_j\partial_j z \tag{1.4}$$

is called the Schrödinger map. We notice that by (1.3) the Schrödinger map is the derivative nonlinear Schrödinger equation of the form

$$i\partial_t z + \Delta z = \frac{2}{1+|z|^2}\,\overline{z}\sum_{j=1}^{n}(\partial_j z)^2. \tag{1.5}$$

We also notice that it is known that the solution of (1.4) preserves the energy $E[z(t)]$ defined by (1.1) (see [7]).

Remark 1.1. The equation (1.5) is also derived from the Heisenberg model of the ferromagnetic spin system

$$\begin{aligned} &u : \mathbf{R}\times\mathbf{R}^n \to S^2 \subset \mathbf{R}^3, \quad n = 1,\, 2,\, 3,\\ &\partial_t u = u \times \Delta u, \end{aligned} \tag{1.6}$$

by using the stereographic projection (see [16]).

Remark 1.2. The Schrödinger map is also formulated in more general setting. Let (N, g, J) be the Riemannian surface with the metric g, complex structure J. Then, the Schrödinger map is described by the map $s : \mathbf{R}\times\mathbf{R}^n \to N$ satisfying

$$\partial_t s = J(s)\sum_{j=1}^{n}\nabla_j\partial_j s, \tag{1.7}$$

where ∇_j denotes the pull-back covariant derivative on $s^{-1}TN$.

1.2 Cauchy problem of the Schrödinger map

In this note, we consider the Cauchy problem of the Schrödinger map

$$\text{(S)} \qquad \begin{cases} \partial_t z = i\displaystyle\sum_{j=1}^{n}\nabla_j\partial_j z, & (t,x)\in(0,\infty)\times\mathbf{R}^n,\\ z|_{t=0} = z_0, & x\in\mathbf{R}^n. \end{cases}$$

In view of to construct the time-global solution to (S), it is natural to treat the class of the initial data the conserved energy $E[z_0]$ is finite. Roughly speaking, such class is $H^1(\mathbf{R}^n)$. So, we first consider the time-local well-posedness of (S) for the data in $H^s(\mathbf{R}^n)$ for small s as long as possible.

The equation in (S) has the scaling invariance. Form the usual scaling argument, it is considered that $s \geq n/2$ is necessary for the local well-posedness of (S). Thus, to treat the local well-posedness of (S) in $H^1(\mathbf{R}^n)$, we must restrict the space dimension n to 1 or 2. The case $n = 2$ is critical in this sense, and in this case we have the following interesting conjecture, which states the geometric structure of the target manifold influences the global behavior of the solution.

- If the target manifold is S^2, then there exists smooth z_0 with $E[z_0] < \infty$ which develops the singularity in finite time.
- If the target manifold is $\mathbf{H}^2$ (the hyperbolic space), then for all smooth z_0 with $E[z_0] < \infty$, (S) has a unique smooth solution globally in time.

As for the Cauchy problem (S), the following results are known. Chang-Shatah-Uhlenbeck [4] showed that when the space dimension $n = 1$, there exists a unique global solution in $H^1(\mathbf{R})$. When the space dimension $n = 2$, they showed the existence of unique small global solution for the data in $H^1(\mathbf{R}^2)$ under the radial or equivariant symmetry assumption. Their method is based on the application of the Hasimoto transformation to transform the Schrödinger map such as (1.5) to the nonlinear Schrödinger equation which have no derivative term in the nonlinearity.

1.3 Gauge transformation

In what follows, we focus to the Schrödinger map from $\mathbf{R} \times \mathbf{R}^2$ to S^2, (1.4) without symmetry assumption. To consider the Schrödinger map for the low regularity data, Nahmod-Stefanov-Uhlenbeck [13] introduced the gauge transformation as follows. For the Schrödinger map z and ∇_α defined by (1.3), we introduce the following transformation

$$u_\alpha = \frac{2}{1+|z|^2}\, e^{i\psi}\partial_\alpha z, \tag{1.8}$$

$$D_\alpha = \frac{2}{1+|z|^2}\, e^{i\psi}\,\nabla_\alpha\,(1+|z|^2)e^{-i\psi} \equiv \partial_\alpha + iA_\alpha, \tag{1.9}$$

for $\alpha = 0, 1, 2$, where ψ is a real-valued function determined later, and $\partial_0 = \partial_t$. Here, we notice that A_α is the real valued function determined by

$$A_\alpha = -\partial_\alpha\psi + \frac{2}{1+|z|^2}\mathrm{Im}(z\,\partial_\alpha\bar{z}).$$

Then, the equation (1.4) is written as

$$u_0 = i\sum_{k=1}^{2} D_k u_k. \tag{1.10}$$

Moreover, the conditions on ∂_α and ∇_α,

$$\nabla_\alpha\partial_\beta z = \nabla_\beta\partial_\alpha z, \quad [\nabla_\alpha, \nabla_\beta] = -4i\,\mathrm{Im}(b_\alpha\bar{b}_\beta),$$

are written as

$$D_\alpha u_\beta = D_\beta u_\alpha, \quad [D_\alpha, D_\beta] = -4i\,\mathrm{Im}(u_\alpha\bar{u}_\beta), \tag{1.11}$$

where $b_j = (1+|z|^2)^{-1}\partial_j z$. In particular, the equations (1.10), (1.11) are invariant for arbitrary choice of the real-valued function ψ in (1.8), (1.9).

Now we apply D_j to the equation (1.10) and we use the conditions (1.11) to obtain

$$D_0 u_j = i\sum_{k=1}^{2} D_k^2\, u_j + \sum_{k=1}^{2} 4\,\mathrm{Im}(u_j\bar{u}_k)u_k. \tag{1.12}$$

Then, we determine ψ by the Coulomb gauge condition

$$\sum_{j=1}^{2}\partial_j A_j = 0, \tag{1.13}$$

which is equivalent to

$$\Delta\psi = \sum_{j=1}^{2}\partial_j\Big\{\frac{2}{1+|z|^2}\mathrm{Im}(z\,\partial_j\bar{z})\Big\}.$$

Such ψ is uniquely determined up to constants for the map z which decays at space infinity. From the condition (1.13) with (1.10), (1.11), A_α, $\alpha = 0, 1, 2$ is determined only by u_j's (see (2.4), (2.6)). The derived system of the nonlinear Schrödinger equations on u_j's is called the modified Schrödinger map (see [13, Theorems 2.1, 2.2]).

Remark 1.3. (1) Since the modified Schrödinger map is derived as the first order derivatives of the original Schrödinger map (see (1.8)), the solution of the modified Schrödinger map in H^s corresponds to the solution to the original Schrödinger map in H^{s+1}.

(2) As is pointed out in [13, §3], it is not possible to go back directly from the solution of the modified Schrödinger map to the original Schrödinger map. However, a priori estimate and the estimate on the time of existence on the smooth solution to (MS) are made use of in order to construct the low regularity solution to the Schrödinger map. See [11, §6] for details.

2 Main Results

The modified Schrödinger map (MS) in two space dimensions is the system of the nonlinear Schrödinger equations of the following form,

$$i\,\partial_t u_1 + \Delta u_1 = -2i\,\boldsymbol{A}\cdot\nabla u_1 + A_0 u_1 + |\boldsymbol{A}|^2 u_1 + 4i\,\mathrm{Im}(u_2\overline{u}_1)u_2, \tag{2.1}$$

$$i\,\partial_t u_2 + \Delta u_2 = -2i\,\boldsymbol{A}\cdot\nabla u_2 + A_0 u_2 + |\boldsymbol{A}|^2 u_2 + 4i\,\mathrm{Im}(u_1\overline{u}_2)u_1, \tag{2.2}$$

$$u_1(0,x) = u_0^1(x), \quad u_2(0,x) = u_0^2(x), \quad x \in \mathbf{R}^2, \tag{2.3}$$

where, $u_j : [0,T]\times\mathbf{R}^2 \ni (t,x) \mapsto u(t,x) \in \mathbf{C}$, $j = 1, 2$ (we set $u = (u_1,u_2)$ in the following), and $\boldsymbol{A} = (A_1[u], A_2[u])$, $A_0 = A_0[u]$ are defined by

$$A_j[u] = 2\,G_j * \mathrm{Im}(u_1\overline{u}_2), \quad j = 1,2, \tag{2.4}$$

$$G_1(x) = \frac{1}{2\pi}\frac{x_2}{|x|^2}, \quad G_2(x) = -\frac{1}{2\pi}\frac{x_1}{|x|^2}, \tag{2.5}$$

$$A_0[u] = -\sum_{j,k=1}^{2} 2R_jR_k\mathrm{Re}(u_j\overline{u}_k) + 2|u|^2. \tag{2.6}$$

Here, $R_j = \partial_j(-\Delta)^{-1/2}$ denotes the Riesz transforms.

From the definition above we have $\mathrm{div}\boldsymbol{A} = 0$. This fact and the fact that $A_j[u]$ is real valued enable us to derive the conservation of the L^2-norm.

Lemma 2.1. *Let u be the solution to (MS). Then, we have*

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad t \geq 0. \tag{2.7}$$

Proof. Multiplying the first equation in (MS) by $\overline{u}_1$ and integrating over $\mathbf{R}^n$, and then taking the imaginary part, we obtain

$$\frac{1}{2}\,\partial_t\|u_1(t)\|_{L^2}^2 = 4\int \mathrm{Im}(u_2\overline{u}_1)\mathrm{Re}(u_2\overline{u}_1)\,dx,$$

since $\operatorname{div} A = 0$, and A_0, A_j is real valued for $j = 1, 2$. Similarly, we obtain

$$\begin{aligned}\frac{1}{2}\,\partial_t\|u_2(t)\|_{L^2}^2 &= 4\int \mathrm{Im}(u_1\overline{u}_2)\mathrm{Re}(u_1\overline{u}_2)\,dx \\ &= -4\int \mathrm{Im}(u_2\overline{u}_1)\mathrm{Re}(u_2\overline{u}_1)\,dx.\end{aligned}$$

Thus, we obtain $\partial_t\|u(t)\|_{L^2}^2 = \partial_t\|u_1(t)\|_{L^2}^2 + \partial_t\|u_2(t)\|_{L^2}^2 = 0$ which implies (2.7). □

Remark 2.2. (1) Due to the relation (1.8), the conservation of the L^2-norm of (MS) corresponds to the conservation of the energy $E[z(t)]$ for the original Schrödinger map.

(2) The modified Schrödinger map is invariant with respect to the scale transformation

$$u(t,x) \mapsto \lambda u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

Then, the scaling argument suggests that the critical space for the local well-posedness of the Cauchy problem (MS) is $L^2(\mathbf{R}^2)$.

As for the modified Schrödinger map, Nahmod-Stefanov-Uhlenbeck [14] showed the existence and uniqueness of the solution for the data $u_0 \in H^s(\mathbf{R}^2)$ with $s > 1$ by using the energy method. In [8], we showed the existence of at least one solution for the data $u_0 \in H^s(\mathbf{R}^2)$ with $s > 1/2$ by using the energy method combined with a variant of the Strichartz estimates. However, uniqueness of solutions was proved only for the data $u_0 \in H^1(\mathbf{R}^2)$ due to the loss of the derivatives in the nonlinearity. In this note, we describe the slightly improved version of the result in [8], especially on the lower bound of the time of the existence (2.8) and the estimate of the solution (2.9).

Theorem 2.3. *Let $u_0 \in H^s(\mathbf{R}^2)$ with $s > 1/2$. Then, there exists $T > 0$ satisfying*

$$\min\bigl\{1, C/\bigl((1 + \|u_0\|_{L^2}^q)\|u_0\|_{H^{1/2+\varepsilon}}^q\bigr)\bigr\} \leq T \leq 1, \tag{2.8}$$

and at least one solution $u \in C_w([0,T];H^s)\cap C([0,T];L^2)$ *to (MS) such that*

$$J^{s-1/2-\varepsilon}u \in L^p(0,T;\ L^q), \tag{2.9}$$

where $J^\delta = (I-\Delta)^{\delta/2}$, $s-1/2>\varepsilon>0$, *and* $1/p = 1/2-1/q$ *with* $2\le q\le\infty$.

Remark 2.4. Similar result have been obtained in Kato [8], Kenig-Nahmod [11]. In Theorem 2.3 we refine such results in the following sense. Firstly, the lower bound of T (2.8) depends on $\|u_0\|_{H^{1/2+\varepsilon}}$ instead of $\|u_0\|_{H^s}$. Secondly, the condition of the solution (2.9) is estimated explicitly. This fact is used effectively to show the uniqueness of the solution in class below H^1 (see [9]).

For the proof of Theorem 2.3 we use the compactness argument. Because the local well-posedness for smooth data is already known (see [14]), our task is to show a priori estimate for the solution to (MS). To recover the loss of the derivatives caused by the nonlinearity, the following type of estimate

$$\|J^s w\|_{L_T^pL_x^q} \lesssim \|w\|_{L_T^\infty H_x^{s+1/2+\varepsilon'}} + \|F\|_{L_T^2H_x^{s-1/2}} \tag{2.10}$$

for the solution to $i\partial_t w + \Delta w = F$ is crucial in our argument, where p, q are the admissible exponent for Strichartz estimates (see Proposition 3.5 below for the precise statement). Compared with the usual Strichartz estimate

$$\|J^s w\|_{L_T^pL_x^q} \lesssim \|w(0)\|_{H^s} + \|F\|_{L_T^1H_x^s},$$

estimate (1.13) says that we have a gain of the regularity $1/2$ on the inhomogeneous term at the cost of a loss of the regularity $1/2+\varepsilon'$ on the homogeneous term. This type of estimate is first appeared in Koch-Tzvetkov [12], and refined by Kenig-Koenig [10] in the context of the Benjamin-Ono equation.

In Theorem 2.3, the uniqueness of the solution is not obtained. As for the uniqueness, the following results have been known. By using the Vladimirov's argument [18] (see also [15]) we obtained the uniqueness of the solution to (MS) in H^1.

Theorem 2.5 ([8]). *Let* $u_0 \in H^1(\mathbf{R}^2)$. *Then, the solution to (MS) in the class of Theorem* 2.3 *is uniquely determined.*

Recently, the condition on the regularity for the uniqueness is improved as follows.

Theorem 2.6 ([9]). *Let $u_0 \in H^s(\mathbf{R}^2)$ with $s > 3/4$. Then, there exists $T > 0$ and unique solution $u \in C([0,T]; H^s)$ to (MS) such that*

$$J^{s-1/2-\varepsilon} u \in L^p(0,T;\ L^q),$$

where $s - 1/2 > \varepsilon > 0$, and $1/p = 1/2 - 1/q$ with $2 \leq q \leq \infty$.

In the rest of this note, we describe the outline of the proof of Theorems 2.3 and 2.6. For simplicity, we consider the following problem

$$\begin{aligned} &i\partial_t u + \Delta u = i\boldsymbol{A}[u] \cdot \nabla u, \quad (t,x) \in (0,T) \times \mathbf{R}^2, \\ &u(0,x) = u_0(x), \quad x \in \mathbf{R}^2, \end{aligned} \tag{2.11}$$

where $\boldsymbol{A}[u] = (A_1[u], A_2[u])$ with

$$A_j[u] = G_j * |u|^2, \quad j = 1, 2. \tag{2.12}$$

This is the essential part of (MS), and it is not hard to handle the full system (MS) to show the same result.

Throughout this note we use the following notation. We denote by $\widehat{f}$ or $\mathcal{F}f$ the Fourier transform of f. We denote $J^s = (I - \Delta)^{s/2}$ and $D^s = (-\Delta)^{s/2}$. H^s is the Sobolev space whose norm is defined by $\|f\|_{H^s} = \|J^s f\|_{L^2}$, and $\dot{H}^s$ is the homogeneous Sobolev space whose semi-norm is defined by $\|f\|_{\dot{H}^s} = \|D^s f\|_{L^2}$. The function space $L^p(0,T;\ L^q(\mathbf{R}^2))$ is simply denoted by $L^p_T L^q_x$, and $L^\infty(0,T;\ H^s(\mathbf{R}^2))$ is also $L^\infty_T H^s_x$.

3 Outline of Proof of Theorem 2.3

In this section, we describe the outline of the proof of Theorem 2.3. In particular, we give a proof of a priori estimate of the solution, which gives the condition (2.9). Once we obtain such a priori estimate, the existence of the solution is similarly proved as in [8].

3.1 Preliminary Estimates

In this subsection we collect the estimates which is used to construct a priori estimates for the solution to (2.11). We first state the usual Strichartz estimates. For the proof, see [2] for example.

Lemma 3.1. *Let $n = 2$. We assume $2 < p \leq \infty$, $2 \leq q < \infty$, and $1/p = 1/2 - 1/q$. Then, the following estimates hold.*

$$\|U(t)f\|_{L^p_T L^q_x} \lesssim \|f\|_{L^2}, \tag{3.1}$$

$$\left\| \int_0^t U(t-t')F(t')dt' \right\|_{L^p_T L^q_x} \lesssim \|F\|_{L^1_T L^2_x}, \tag{3.2}$$

where $U(t) = e^{it\Delta}$.

Lemma 3.2. *For $s > 0$, $1 < p < \infty$, we have*

$$\|D^s(fg)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|D^s g\|_{L^{p_2}} + \|g\|_{L^{r_1}} \|D^s f\|_{L^{r_2}},$$
$$\|J^s(fg)\|_{L^p} \lesssim \|f\|_{L^{p_1}} \|J^s g\|_{L^{p_2}} + \|g\|_{L^{r_1}} \|J^s f\|_{L^{r_2}},$$

where $1/p = 1/p_1 + 1/p_2 = 1/r_1 + 1/r_2$ with $p < p_1, r_1 \leq \infty$.

Proof. See [3, Proposition 1.2], for example. Note that $\|f\|_{\dot{F}^s_{p,2}} \simeq \|D^s f\|_{L^p}$ and $\|f\|_{F^s_{p,2}} \simeq \|J^s f\|_{L^p}$ for $1 < p < \infty$. □

Below we collect the estimates on the potential term $\boldsymbol{A}[u]$ defined by (2.12). Of course, the same estimate also holds $\boldsymbol{A}[u]$ defined by (2.4).

Lemma 3.3. *We assume $s \geq 0$, $1 > \delta > 2/q > 0$, and $2 \leq p < \infty$ if $s > 0$; $2 < p < \infty$ if $s = 0$. Then, we have*

$$\|\nabla \boldsymbol{A}[u]\|_{\dot{H}^s} \lesssim \|u\|_{L^\infty} \|u\|_{\dot{H}^s}, \tag{3.3}$$
$$\|\nabla \boldsymbol{A}[u]\|_{L^\infty} \lesssim \|u\|_{L^\infty} \|J^\delta u\|_{L^q}, \tag{3.4}$$
$$\|\boldsymbol{A}[u]\|_{L^\infty} \lesssim \|u\|_{L^2} \|J^\delta u\|_{L^q}, \tag{3.5}$$
$$\|D^s \boldsymbol{A}[u]\|_{L^p} \lesssim \|u\|_{L^p} \|u\|_{\dot{H}^s}. \tag{3.6}$$

Proof. Since $\boldsymbol{A}[u]$ is given by

$$\boldsymbol{A}[u] = (-\Delta)^{-1} \mathrm{rot}\, |u|^2,$$

it is not hard to show the above estimates by using Lemma 3.2 and the Sobolev embedding. See [8] for the detail. □

The following is the energy estimate for the solution to (2.11).

Proposition 3.4. *Let $s \geq 0$ and let u be a solution to* (2.11) *on* $(0,T) \times \mathbf{R}^2$*. Then, for $T > 0$, we have*

$$\|u\|_{L_T^\infty H_x^s} \leq \|u_0\|_{H^s} \exp\{C(1+\|u_0\|_{L^2}^2)\,T^{2/q}\|J^\delta u\|_{L_T^p L_x^q}^2\}, \tag{3.7}$$

where $\delta > 2/q > 0$, $1/p = 1/2 - 1/q$.

Proof. For the proof of the energy estimate (3.7), we employ [14, Propositions 2] which states

$$\partial_t \|u(t)\|_{\dot H^s}^2 \lesssim \left(\|\nabla \boldsymbol{A}(t)\|_{\dot H^s}\|u(t)\|_{L^\infty} + \|\nabla \boldsymbol{A}(t)\|_{L^\infty}\|u(t)\|_{\dot H^s}\right)\|u(t)\|_{\dot H^s}. \tag{3.8}$$

This estimate is proved by using the commutator estimate combined with the Littlewood-Paley decomposition. In what follows, we estimate each term on the right hand side of (3.8) to obtain (3.7) assuming $s > 0$.

From (3.3) we have

$$\|\nabla \boldsymbol{A}(t)\|_{\dot H^s}\|u(t)\|_{L^\infty} \lesssim \|u(t)\|_{L^\infty}^2\|u(t)\|_{\dot H^s} \lesssim \|J^\delta u(t)\|_{L^q}^2\|u(t)\|_{\dot H^s} \tag{3.9}$$

by the Sobolev embedding, where $\delta > 2/q$. Similarly, from (3.4) we have

$$\begin{aligned}\|\nabla \boldsymbol{A}(t)\|_{L^\infty}\|u(t)\|_{\dot H^s} &\lesssim \|u(t)\|_{L^\infty}\|J^\delta u(t)\|_{L^q}\|u(t)\|_{\dot H^s}\\ &\lesssim \|J^\delta u(t)\|_{L^q}^2\|u(t)\|_{\dot H^s}.\end{aligned} \tag{3.10}$$

Thus, we obtain

$$\partial_t \|u(t)\|_{\dot H^s}^2 \leq C(1+\|u_0\|_{L^2}^2)\|J^\delta u(t)\|_{L^q}^2\|u(t)\|_{\dot H^s}^2.$$

Now we apply the Gronwall inequality to obtain

$$\|u(t)\|_{\dot H^s} \leq \|u_0\|_{\dot H^s}\exp\Big(C(1+\|u_0\|_{L^2}^2)\int_0^T \|J^\delta u(t')\|_{L^q}^2 dt'\Big).$$

Therefore, the conservation of the L^2-norm (see Lemma 2.1) and the Hölder inequality with respect to the time integral enables us to obtain (3.7). □

We finally state the crucial estimate for the proof of Theorem 2.3. This type of estimate was first given by Koch-Tzvetkov [12] and refined by Kenig-Koenig [10] in the context of the Benjamin-Ono equation.

Proposition 3.5. *Let $T \le 1$. We assume that w is a solution to the equation*

$$i\partial_t w + \Delta w = F, \quad (t,x) \in (0,T) \times \mathbf{R}^2. \tag{3.11}$$

Then, for $s \in \mathbf{R}$, $\varepsilon' > 0$, we have

$$\|J^s w\|_{L_T^p L_x^q} \lesssim \|w\|_{L_T^\infty H^{s+1/2+\varepsilon'}} + \|F\|_{L_T^2 H_x^{s-1/2}}, \tag{3.12}$$

where $1/p = 1/2 - 1/q$, $2 \le q < \infty$.

Proof. The proof is essentially due to [10, Proposition 2.8]. We first introduce the Littlewood-Paley decomposition. Let $\varphi \in C_0^\infty(\mathbf{R}^2)$ satisfy $\varphi(\xi) = 1$ for $|\xi| \le 1/2$, $\varphi(\xi) = 0$ for $|\xi| \ge 1$. And we set $\eta(\xi) = \varphi(\xi/2) - \varphi(\xi)$ and set $\eta_k(\xi) = \eta(\xi/2^k)$ for $k \ge 0$ so that $\operatorname{supp} \eta_k \subset \{2^{k-1} \le |\xi| \le 2^{k+1}\}$ and $1 = \varphi(\xi) + \sum_{k=0}^\infty \eta_k(\xi)$. Then, we define Δ_k by $\widehat{\Delta_k f} = \eta_k \widehat{f}$ and S_0 by $\widehat{S_0 f} = \varphi \widehat{f}$, where $\widehat{g}$ denotes the Fourier transform of g. Using the notation above, it is known that

$$\|f\|_{L^r(\mathbf{R}^n)} \simeq \left\| \left(|S_0 f|^2 + \sum_{k=0}^\infty |\Delta_k f|^2 \right)^{1/2} \right\|_{L^r(\mathbf{R}^n)}$$

holds for $1 < r < \infty$.

Since $2 \le q < \infty$, using the Littlewood-Paley decomposition of $J^s w$ we have

$$\begin{aligned} \|J^s w\|_{L_T^p L_x^q} &\lesssim \left\| \left(|S_0 J^s w|^2 + \sum_{k=0}^\infty |\Delta_k J^s w|^2 \right)^{1/2} \right\|_{L_T^p L_x^q} \\ &\lesssim \left(\|J^s S_0 w\|_{L_T^p L_x^q}^2 + \sum_{k=0}^\infty \|J^s \Delta_k w\|_{L_T^p L_x^q}^2 \right)^{1/2}. \end{aligned}$$

For the last inequality above we used the Minkowski's integral inequality, since $p, q \ge 2$.

Before applying Strichartz estimates to estimate $\|J^s \Delta_k w\|_{L_T^p L_x^q}^2$, we prepare the disjoint decomposition of the time interval $[0,T] = \cup_{j=1}^m I_j$, where $I_j = [a_j, a_{j+1})$ satisfy $|I_j| = 2^{-k}$ for $1 \le j \le m-1$ and $2^{-k} \le |I_m| \le 2^{-k+1}$. Then, we have $m \le 2^k$, since $(m-1)2^{-k} + |I_m| = T$ implies $2^{-k} m \le T - 2^{-k} + 2^{-k} = T \le 1$. Now we apply the decomposition as follows.

$$\begin{aligned}\|J^s\Delta_k w\|_{L_T^pL_x^q} &\lesssim 2^{sk}\|\Delta_k w\|_{L_T^pL_x^q}\\ &= 2^{sk}\Big(\sum_{j=1}^m \|\Delta_k w\|_{L_{I_j}^pL_x^q}^p\Big)^{1/p}\\ &\lesssim 2^{sk}\Big(\sum_{j=1}^m \|\Delta_k w\|_{L_{I_j}^pL_x^q}^2\Big)^{1/2},\end{aligned}$$

since $l^2 \hookrightarrow l^p$ for $p > 2$. Since $\Delta_k w$ satisfies the following integral equation

$$\Delta_k w(t) = U(t)\Delta_k w(a_j) - i\int_{a_j}^t U(t-t')\Delta_k F(t')\,dt' \tag{3.13}$$

for $t \in I_j$, applying Lemma 3.1 we have

$$\|\Delta_k w\|_{L_{I_j}^pL_x^q} \lesssim \|\Delta_k w(a_j)\|_{L^2}^2 + \|\Delta_k F\|_{L_{I_j}^1L_x^2}.$$

Thus, we obtain

$$\begin{aligned}&2^{sk}\Big(\sum_{j=1}^m \|\Delta_k w\|_{L_{I_j}^pL_x^q}^2\Big)^{1/2}\\ &\lesssim 2^{sk}\Big\{\sum_{j=1}^m\big(\|\Delta_k w(a_j)\|_{L^2}^2 + \|\Delta_k F\|_{L_{I_j}^1L_x^2}^2\big)\Big\}^{1/2}\\ &\lesssim 2^{sk}\Big\{m\|\Delta_k w\|_{L_T^\infty L_x^2}^2 + \sum_{j=1}^m |I_j|\|\Delta_k F\|_{L_{I_j}^2L_x^2}^2\Big\}^{1/2}\\ &\lesssim 2^{(s+1/2)k}\|\Delta_k w\|_{L_T^\infty L_x^2} + 2^{(s-1/2)k}\|\Delta_k F\|_{L_T^2L_x^2}\\ &\lesssim 2^{-\varepsilon' k}\|w\|_{L_T^\infty H_x^{s+1/2+\varepsilon'}} + \|\Delta_k J^{s-1/2}F\|_{L_T^2L_x^2}.\end{aligned}$$

For the last term in the second inequality above, we used the triangle inequality in l^2 and then applied $l^1 \hookrightarrow l^2$. Meanwhile, applying Lemma 3.1 it is easy to see that

$$\begin{aligned}\|J^sS_0w\|_{L_T^pL_x^q} &\lesssim \|S_0w(0)\|_{L^2} + \|S_0F_1\|_{L_T^1L_x^2} + \|S_0F_2\|_{L_T^1L_x^2}\\ &\lesssim \|w\|_{L_T^\infty L_x^2} + \|S_0J^{s-1/2}F_1\|_{L_T^2L_x^2} + \|S_0J^sF_2\|_{L_T^1L_x^2}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\Big(\|J^s S_0 w\|^2_{L^p_T L^q_x} + \sum_{k=0}^{\infty} \|J^s \Delta_k w\|^2_{L^p_T L^q_x}\Big)^{1/2} \\
&\lesssim \Big(\sum_{k=0}^{\infty} 2^{-2\varepsilon' k} \|w\|^2_{L^\infty_T H^{s+1/2+\varepsilon'}_x}\Big)^{1/2} + \Big(\|S_0 J^{s-1/2} F\|^2_{L^2_T L^2_x} + \sum_{k=0}^{\infty} \|\Delta_k J^{s-1/2} F\|^2_{L^2_T L^2_x}\Big)^{1/2} \\
&\lesssim \Big(\sum_{k=0}^{\infty} 2^{-2\varepsilon' k}\Big)^{1/2} \|w\|_{L^\infty_T H^{s+1/2+\varepsilon'}_x} + \Big\|\Big(|S_0 J^{s-1/2} F|^2 + \sum_{k=0}^{\infty} |\Delta_k J^{s-1/2} F|^2\Big)^{1/2}\Big\|_{L^2_T L^2_x} \\
&\lesssim \|w\|_{L^\infty_T H^{s+1/2+\varepsilon'}_x} + \|J^{s-1/2} F_1\|_{L^2_T L^2_x},
\end{aligned}$$

where we applied the Minkowski's integral inequality again for the second inequality above. This completes the proof of Proposition 3.5. $\square$

3.2 A Priori Estimate

Below we show a priori estimate to the solution to (2.11). Once we obtain such a priori estimate, Theorem 2.3 is similarly proved as in [8] by using the compactness argument.

Theorem 3.6. *We assume $s > 1/2$ and choose $\varepsilon > 0$ satisfying $s - 1/2 > \varepsilon$. Let u be a smooth solution to* (2.11). *Then, for any $\varepsilon > 0$, there exists T satisfying*

$$\min\{1, C/((1 + \|u_0\|^q_{L^2})\|u_0\|^q_{H^{1/2+\varepsilon}})\} \leq T \leq 1$$

and $M > 0$ such that

$$\|J^\delta u\|_{L^p_T L^q_x} \leq M \|u_0\|_{H^{1/2+\varepsilon}}, \tag{3.14}$$

where $\varepsilon > \delta > 2/q > 0$, $1/p = 1/2 - 1/q$. Moreover, we have

$$\|J^{s-1/2-\varepsilon} u\|_{L^p_T L^q_x} \leq C(\|u_0\|_{H^s}). \tag{3.15}$$

Proof. Applying Proposition 3.5 for $F = i\boldsymbol{A}[u] \cdot \nabla u$ with $s = \delta$, we obtain

$$\|J^\delta u\|_{L^p_T L^q_x} \lesssim \|u\|_{L^\infty_T H^{1/2+\varepsilon}_x} + \|\boldsymbol{A}[u] \cdot \nabla u\|_{L^2_T H^{-1/2+\delta}_x} \tag{3.16}$$

where we substituted $\delta + \varepsilon' = \varepsilon$. In what follows we estimate each term on the right hand side of (3.16) to obtain (3.14).

The first term is easily estimated by applying energy estimate (3.7) directly,

$$\|u\|_{L_T^\infty H_x^{1/2+\varepsilon}} \le \|u_0\|_{H^{1/2+\varepsilon}} \exp\{C(1+\|u_0\|_{L^2}^2)\, T^{2/q}\|J^\delta u\|_{L_T^p L_x^q}^2\}.$$

As for the second term we notice that $\boldsymbol{A}[u]\cdot\nabla u = \operatorname{div}(\boldsymbol{A}[u]\,u)$, since $\operatorname{div}\boldsymbol{A}[u]=0$. Using this, we have

$$\begin{aligned}\|\boldsymbol{A}[u]\cdot\nabla u\|_{L_T^2 H_x^{-1/2+\delta}} &\lesssim \|D^{1/2+\delta}(\boldsymbol{A}[u]\,u)\|_{L_T^2 L_x^2}\\ &\lesssim \|u\|_{L_T^2 L_x^\infty}\|D^{1/2+\delta}\boldsymbol{A}[u]\|_{L_T^\infty L_x^2}\\ &\quad + \|\boldsymbol{A}[u]\|_{L_T^2 L_x^\infty}\|D^{1/2+\delta}u\|_{L_T^\infty L_x^2}.\end{aligned}$$

Then, applying (3.5), (3.6), we have

$$\begin{aligned}\|D^{1/2+\delta}\boldsymbol{A}[u]\|_{L_T^\infty L_x^2} &\lesssim \|u\|_{L_T^\infty L_x^2}\|D^{1/2+\delta}u\|_{L_T^\infty L_x^2},\\ \|\boldsymbol{A}[u]\|_{L_T^2 L_x^\infty} &\lesssim \|u\|_{L_T^\infty L_x^2}\|J^\delta u\|_{L_T^2 L_x^q}.\end{aligned}$$

Thus, from conservation of the L^2-norm and the energy estimate (3.7), we obtain

$$\begin{aligned}\|\boldsymbol{A}[u]\cdot\nabla u\|_{L_T^2 H_x^{-1/2+\delta}} &\lesssim \|u_0\|_{L_x^2}\|J^\delta u\|_{L_T^2 L_x^q}\|u\|_{L_T^\infty H^{1/2+\delta}}\\ &\lesssim (1+\|u_0\|_{L^2}^2\|J^\delta u\|_{L_T^2 L_x^q}^2)\|u\|_{L_T^\infty H^{1/2+\delta}}\\ &\lesssim \|u_0\|_{H^{1/2+\delta}}\exp\{C(1+\|u_0\|_{L^2}^2)\, T^{2/q}\|J^\delta u\|_{L_T^p L_x^q}^2\}.\end{aligned}$$

Therefore, we obtain

$$\|J^\delta u\|_{L_T^p L_x^q} \le C_0\|u_0\|_{H^{1/2+\varepsilon}}\exp\{C_1(1+\|u_0\|_{L^2}^2)\, T^{2/q}\|J^\delta u\|_{L_T^p L_x^q}^2\}. \tag{3.17}$$

Now we set $K(T)=\|J^\delta u\|_{L_T^p L_x^q}^2$. Then, $K(T)$ is a continuous function with respect to T since $2\le p<\infty$, and (3.17) implies

$$K(T) \le C_0^2\|u_0\|_{H^{1/2+\varepsilon}}^2\exp\{2C_1(1+\|u_0\|_{L^2}^2)\, T^{2/q}K(T)\}. \tag{3.18}$$

If $K(T)\le C_0^2 e\,\|u_0\|_{H^{1/2+\varepsilon}}^2$ holds for $0\le T\le 1$, then the conclusion of Theorem 3.6 follows. On the other hand, in the case of there exists $T_1\in(0,1)$ such that $K(T_1) > C_0^2 e\,\|u_0\|_{H^{1/2+\varepsilon}}^2$, we set

$$T_0 = \inf\{T>0;\ K(T) > C_0^2 e\,\|u_0\|_{H^{1/2+\varepsilon}}^2\}.$$

Then, $T_0>0$ and we have $K(T_0) = C_0^2 e\,\|u_0\|_{H^{1/2+\varepsilon}}^2$. Thus, (3.18) with $T=T_0$ implies that

$$e \le \exp\{2C_1(1+\|u_0\|_{L^2}^2)\, T_0^{2/q} C_0^2 e\,\|u_0\|_{H^{1/2+\varepsilon}}^2\}.$$

Therefore, we obtain the lower bound of T_0,

$$T_0 \geq \frac{1}{(2C_0^2 C_1 e)^{q/2}(1+\|u_0\|_{L^2}^q)\|u_0\|_{H^{1/2+\varepsilon}}^q},$$

and for $0 \leq T \leq T_0$, $K(T) \leq K(T_0) = C_0^2 e\,\|u_0\|_{H^{1/2+\varepsilon}}^2$ holds.

Finally, we prove a priori estimate (3.15). We apply Proposition 3.5 again for $F = i\boldsymbol{A}[u]\cdot\nabla u$ to obtain

$$\|J^{s-1/2-\varepsilon}u\|_{L_T^p L_x^q} \lesssim \|u\|_{L_T^\infty H_x^s} + \|\boldsymbol{A}[u]\cdot\nabla u\|_{L_T^2 H_x^{s-\varepsilon}}. \tag{3.19}$$

Since the first term is also estimated applying the energy estimate (3.7),

$$\|u\|_{L_T^\infty H_x^s} \leq \|u_0\|_{H^s}\exp\{C(1+\|u_0\|_{L^2}^2)\,T^{2/q}\|J^\delta u\|_{L_T^p L_x^q}^2\},$$

it suffices to estimate the second term on the right hand side of (3.19) to obtain (3.15). The second term is also estimated similarly as before,

$$\begin{aligned}\|\boldsymbol{A}[u]\cdot\nabla u\|_{L_T^2 H_x^{s-\varepsilon}} &\lesssim \|D^{s-\varepsilon}(\boldsymbol{A}[u]\,u)\|_{L_T^2 L_x^2}\\ &\lesssim \|u_0\|_{L_x^2}\|J^\delta u\|_{L_T^2 L_x^q}\|u\|_{L_T^\infty H^{s-\varepsilon}}\\ &\lesssim \|u_0\|_{H^{s-\varepsilon}}\exp\{C(1+\|u_0\|_{L^2}^2)\,T^{2/q}\|J^\delta u\|_{L_T^p L_x^q}^2\}.\end{aligned}$$

Thus, by using (3.14) we obtain

$$\begin{aligned}\|J^{s-1/2-\varepsilon}u\|_{L_T^p L_x^q} &\leq C_0\|u_0\|_{H^s}\exp\{C_1(1+\|u_0\|_{L^2}^2)\,T^{2/q}\|J^\delta u\|_{L_T^p L_x^q}^2\}\\ &\leq C_0\|u_0\|_{H^s}\exp\{C_1(1+\|u_0\|_{L^2}^2)\,T^{2/q}M\|u_0\|_{H^{1/2+\varepsilon}}\}.\end{aligned}$$

This completes the proof of (3.15). $\square$

4 Outline of Proof of Theorem 2.6

In this section we describe the idea of the proof of Theorem 2.6. For the proof, it suffices to show the following theorem.

Theorem 4.1 ([9]). *Let u and v be smooth solutions to (MS) with the same smooth data satisfying*

$$u, v \in L^\infty(0,T;H^{1/2}) \cap L^p(0,T;B_{q,2}^{1/2}) \tag{4.1}$$

for some $q > 4$ with $1/p = 1/2 - 1/q$. Then, $u \equiv v$ holds. Moreover, the estimate

$$\|u(t) - v(t)\|_{H^{-1/2}} \leq C\|u(t') - v(t')\|_{H^{-1/2}} \tag{4.2}$$

holds when $t > t'$, where the constant C depends on $\|u\|_{L^\infty_T H^{1/2}}$, $\|v\|_{L^\infty_T H^{1/2}}$, $\|u\|_{L^p_T B^{1/2}_{q,2}}$, and $\|v\|_{L^p_T B^{1/2}_{q,2}}$, and $B^s_{p,q}$ is the Besov space.

Let u, v be the solutions to (P), then $w \equiv u - v$ satisfy

$$i\,\partial_t w + \Delta w = i\boldsymbol{A}[u] \cdot \nabla w + i(\boldsymbol{A}[u] - \boldsymbol{A}[v]) \cdot \nabla v. \tag{4.3}$$

The usual way to show the uniqueness is to estimate the L^2-norm of w. In fact, multiplying $\overline{w}$ to both sides of the equation (4.3), taking the imaginary part, and then integrating over $\mathbf{R}^2$, we obtain

$$\frac{1}{2}\,\partial_t \|w(t)\|^2_{L^2} = \mathrm{Re} \int_{\mathbf{R}^2} (\boldsymbol{A}[u] - \boldsymbol{A}[v]) \cdot \nabla v\,\overline{w}\,dx.$$

If we consider the solutions in the class

$$u, v \in C([0, T]; H^s)$$

with $s > 1$, then the uniqueness of solutions is easily obtained as follows. Let $1 < s_0 < \min(s, 2)$, and set $1/p = 1 - s_0/2$, $1/2 = 1/p + 1/q$, and $1/r = 1/q + 1/2$. Then, applying the Hölder inequality and the Sobolev embedding we obtain

$$\begin{aligned}
\frac{1}{2}\,\partial_t \|w(t)\|^2_{L^2} &\leq \left| \int_{\mathbf{R}^2} (\boldsymbol{A}[u] - \boldsymbol{A}[v]) \cdot \nabla v\,\overline{w}\,dx \right| \\
&\leq \|D^{-1}(|u|^2 - |v|^2)\|_{L^q} \|\nabla v\|_{L^p} \|w\|_{L^2} \\
&\lesssim \||u|^2 - |v|^2\|_{L^r} \|v\|_{\dot{H}^{s_0}} \|w\|_{L^2} \\
&\lesssim (\|u\|_{L^q} + \|v\|_{L^q}) \|v\|_{\dot{H}^{s_0}} \|w\|^2_{L^2} \\
&\lesssim (\|u\|_{\dot{H}^{2-s_0}} + \|v\|_{\dot{H}^{2-s_0}}) \|v\|_{\dot{H}^{s_0}} \|w\|^2_{L^2}.
\end{aligned}$$

Since $H^s \hookrightarrow \dot{H}^{2-s_0}$, $H^s \hookrightarrow \dot{H}^{s_0}$, by using the Gronwall inequality we obtain

$$\|w(t)\|_{L^2} \leq C\|w(0)\|_{L^2},$$

which implies the uniqueness of solutions.

To show the uniqueness of less regular solutions, we consider the estimate of w in $H^{-1/2}$ instead of L^2 to overcome the loss of the derivative the nonlinearity. We use the following energy estimate.

Lemma 4.2. *Let w be a solution to*

$$i\partial_t w + \Delta w - i\boldsymbol{a}\cdot\nabla w = F, \tag{4.4}$$

where $\boldsymbol{a}$ is $\mathbf{R}^2$-valued function. Then, for $0 < s < 1$, $0 < t < T$, we have

$$\|w(t)\|_{H^{-s}} \le \exp\Big\{C\int_0^t \|\nabla\boldsymbol{a}(t')\|_{L^\infty}dt'\Big\}\Big(\|w(0)\|_{H^{-s}} + \int_0^t \|F(t')\|_{H^{-s}}dt'\Big). \tag{4.5}$$

Idea of Proof of Lemma 4.2. For $0 \le \tau < T$, we denote by $S(t,\tau)f$ the solution to

$$\begin{cases} i\,\partial_t v + \Delta v - i\boldsymbol{a}\cdot\nabla v = 0, & (t,x)\in(\tau,T)\times\mathbf{R}^2,\\ v(\tau,x) = f(x), & x\in\mathbf{R}^2.\end{cases}$$

Then, the solution to (4.4) is written as

$$w(t) = S(t,0)w(0) - i\int_0^t S(t,\tau)F(\tau)d\tau.$$

Thus, to prove (4.5) it suffices to show

$$\|S(t,\tau)f\|_{H^{-s}} \le \exp\Big\{C\int_0^t \|\nabla\boldsymbol{a}(t')\|_{L^\infty}dt'\Big\}\|f\|_{H^{-s}}. \tag{4.6}$$

To prove (4.6) we consider the dual problem for fixed $t\in(0,T]$,

$$\begin{cases} i\,\partial_\tau \tilde{v} + \Delta\tilde{v} - i\nabla\cdot(\boldsymbol{a}\,\tilde{v}) = 0, & (\tau,x)\in(0,t)\times\mathbf{R}^2,\\ \tilde{v}(t,x) = g(x), & x\in\mathbf{R}^2.\end{cases}$$

We denote by $\tilde{S}(\tau,t)g$ the solution to the problem above. Then, $\tilde{S}(\tau,t)$ is dual operator to $S(t,\tau)$. In fact, the simple calculation shows that

$$\partial_{t'}\langle S(t',\tau)f, \tilde{S}(t',t)g\rangle = 0$$

by using the equation, and integrating this from τ to t we derive

$$\langle S(t,\tau)f, g\rangle = \langle f, \tilde{S}(\tau,t)g\rangle.$$

Meanwhile, from the equation we have

$$\partial_\tau\|\tilde{v}(\tau)\|_{L^2}^2 \le \int \boldsymbol{a}\cdot\nabla|\tilde{v}|^2dx \le \|\nabla\boldsymbol{a}\|_{L^\infty}\|\tilde{v}(\tau)\|_{L^2}^2.$$

Similarly, we have

$$\partial_\tau\|\nabla\tilde{v}(\tau)\|_{L^2} \le C\|\nabla\boldsymbol{a}\|_{L^\infty}\|\nabla\tilde{v}(\tau)\|_{L^2}.$$

Thus, interpolating them we obtain

$$\|\tilde{S}(\tau,t)g\|_{H^s} \le \exp\Big\{C\int_0^t \|\nabla \boldsymbol{a}(t')\|_{L^\infty}dt'\Big\}\|g\|_{H^s}, \tag{4.7}$$

for $0 \le s \le 1$. Therefore, by using the duality we obtain

$$\begin{aligned}
\|S(t,\tau)f\|_{H^{-s}} &= \sup_{\|\varphi\|_{H^s}=1}\Big|\int S(t,\tau)f\,\varphi\,dx\Big| \\
&= \sup_{\|\varphi\|_{H^s}=1}\Big|\int f\,\tilde{S}(\tau,t)\varphi\,dx\Big| \\
&\le \sup_{\|\varphi\|_{H^s}=1}\|f\|_{H^{-s}}\|\tilde{S}(\tau,t)\varphi\|_{H^s} \\
&\le \exp\Big\{C\int_0^t \|\nabla \boldsymbol{a}(\tau)\|_{L^\infty}d\tau\Big\}\|f\|_{H^{-s}}.
\end{aligned}$$

Thus we obtain (4.6). □

Applying Lemma 4.2 to (4.3) with $s = 1/2$ we obtain

$$\begin{aligned}
\|w(t)\|_{H^{-1/2}} \le \exp&\Big\{C\int_0^T \|\nabla \boldsymbol{A}[u](t')\|_{L^\infty}dt'\Big\} \\
&\times\Big(\|w(0)\|_{H^{-1/2}} + \int_0^t \|(\boldsymbol{A}[u]-\boldsymbol{A}[v])\cdot\nabla v\|_{H^{-1/2}}dt'\Big).
\end{aligned} \tag{4.8}$$

Since $\nabla \boldsymbol{A}[u] \sim R_jR_k|u|^2$, for sufficiently small $\delta > 0$ and $\delta > 2/\tilde{q}$, we have

$$\|\nabla \boldsymbol{A}[u]\|_{L^\infty} \lesssim \|J^\delta R_jR_k|u|^2\|_{L^{\tilde{q}}} \lesssim \|J^\delta|u|^2\|_{L^{\tilde{q}}} \lesssim \|J^\delta u\|_{L^q}^2 \lesssim \|u\|_{B^{1/2}_{q,2}}^2.$$

So, the problem is to estimate the product of functions in the Sobolev spaces of negative order which appears in the last term in (4.8).

Remark 4.3. One might think there would be another possibility to apply Lemma 4.2 instead of $H^{-1/2}$. However, from the general version of the lemma below, and from the structure of the nonlinear term, the space $H^{-1/2}$ provides the best result in our method.

Lemma 4.4. *Suppose $n = 2$ and $q > 4$. Then the following estimates hold.*

$$\|fg\|_{H^{-1/2}} \lesssim \|g\|_{B^{1/2}_{q,2}}\|f\|_{H^{-1/2}}, \tag{4.9}$$

$$\|(\boldsymbol{G}*(fg))\nabla h\|_{H^{-1/2}} \lesssim \big(\|g\|_{H^{1/2}}\|h\|_{H^{1/2}} + \|g\|_{B^{1/2}_{q,2}}\|h\|_{B^{1/2}_{q,2}}\big)\|f\|_{H^{-1/2}}. \tag{4.10}$$

If we apply (4.10) to estimate the last term of (4.8), then we obtain

$$\|w(t)\|_{H^{-1/2}} \le C\Big(\|w(0)\|_{H^{-1/2}} + \int_0^t (\|u(\tau)\|_X^2 + \|v(\tau)\|_X^2)\|w(\tau)\|_{H^{-1/2}} d\tau\Big),$$

where we denoted $X = H^{1/2} \cap B^{1/2}_{q,2}$. Thus, by the Gronwall inequality we obtain

$$\|w(t)\|_{H^{-1/2}} \le C\|w(t)\|_{H^{-1/2}}.$$

Thus, Theorem 4.1, the uniqueness of the solution, follows.

Finally we describe the idea of the proof of Lemma 4.4.

Idea of Proof of Lemma 4.4. To prove (4.10) we first show that

$$\|fg\|_{H^{1/2}} \lesssim \|g\|_{B^{1/2}_{q,2}} \|f\|_{H^{1/2}} \tag{4.11}$$

holds. In fact, by using fractional Leibniz rule we have

$$\|fg\|_{H^{1/2}} \lesssim \|f\|_{H^{1/2}} \|g\|_{B^0_{\infty,2}} + \|f\|_{B^0_{r,2}} \|g\|_{B^{1/2}_{q,2}},$$

where $1/2 = 1/q + 1/r$. Then, the embeddings $B^{1/2}_{q,2} \hookrightarrow B^0_{\infty,2}$ and $H^{1/2} \hookrightarrow H^{2/q} \hookrightarrow B^0_{r,2}$ give (4.11). Thus, by using the duality we obtain

$$\begin{aligned} \|fg\|_{H^{-1/2}} &= \sup_{\|\varphi\|_{H^{1/2}}=1} \Big| \int fg\varphi\, dx \Big| \\ &\le \sup_{\|\varphi\|_{H^{1/2}}=1} \|f\|_{H^{-1/2}} \|g\varphi\|_{H^{1/2}} \\ &\lesssim \|f\|_{H^{-1/2}} \|g\|_{B^{1/2}_{q,2}}. \end{aligned}$$

Now we turn to the proof of (4.10). Since $\operatorname{div} \boldsymbol{G} * (fg) = 0$, we have

$$\|(\boldsymbol{G} * (fg))\nabla h\|_{H^{-1/2}} = \|\operatorname{div}\{(\boldsymbol{G} * (fg))h\}\|_{H^{-1/2}} \lesssim \|(\boldsymbol{G} * (fg))h\|_{H^{1/2}}. \tag{4.12}$$

To estimate the right hand side of (4.12) we divide $\boldsymbol{G} * (fg)$ into the high frequency part and the low frequency part,

$$\boldsymbol{G} * (fg) = S_0(\boldsymbol{G} * (fg)) + (1 - S_0)(\boldsymbol{G} * (fg)). \tag{4.13}$$

Here, S_0 is defined as the Fourier multiplier by φ, where $\varphi \in C_0^\infty(\mathbf{R}^2)$ with $\varphi \equiv 1$ near the origin.

As for the high frequency part, the second term on the right hand side of (4.13), we easily obtain

$$\begin{aligned}\|\{(1-S_0)(\boldsymbol{G}*(fg))\}h\|_{H^{1/2}} &\lesssim \|h\|_{B^{1/2}_{q,2}}\|(1-S_0)(\boldsymbol{G}*(fg))\|_{H^{1/2}}\\ &\lesssim \|h\|_{B^{1/2}_{q,2}}\|fg\|_{H^{-1/2}}\\ &\lesssim \|h\|_{B^{1/2}_{q,2}}\|g\|_{B^{1/2}_{q,2}}\|f\|_{H^{-1/2}},\end{aligned}$$

by using (4.11), (4.9).

As for the low frequency part, the first term on the right hand side of (4.13), we estimate

$$\|\{S_0(\boldsymbol{G}*(fg))\}h\|_{H^{1/2}} \lesssim \|S_0(\boldsymbol{G}*(fg))\|_{W^{1,\infty}}\|h\|_{H^{1/2}}. \tag{4.14}$$

To complete the proof we have to estimate $S_0(\boldsymbol{G}*(fg))$ and its gradient. By translation invariance it suffices to do this at the origin. The argument for $S_0(\boldsymbol{G}*(fg))$ and for its gradient is the same. We observe that

$$\begin{aligned}S_0(\boldsymbol{G}*(fg))(0) &= \{\boldsymbol{\Phi}*(fg)\}(0)\\ &= \int_{\mathbf{R}^2}\boldsymbol{\Phi}(y)f(y)g(y)dx,\end{aligned}$$

where we set $\boldsymbol{\Phi} = \mathcal{F}^{-1}[\varphi]*\boldsymbol{G}$. Note that $\boldsymbol{\Phi} \in L^r(\mathbf{R}^2)$ for $2 < r < \infty$. Thus,

$$\begin{aligned}|S_0(\boldsymbol{G}*(fg))(0)| &= \left|\int_{\mathbf{R}^2}\boldsymbol{\Phi}(y)f(y)g(y)dy\right|\\ &\leq \|\boldsymbol{\Phi}\, g\|_{H^{1/2}}\|f\|_{H^{-1/2}}\\ &\lesssim \|\boldsymbol{\Phi}\|_{B^{1/2}_{q,2}}\|g\|_{H^{1/2}}\|f\|_{H^{-1/2}}.\end{aligned}$$

Finally, we notice that

$$\|\boldsymbol{\Phi}\|_{B^{1/2}_{q,2}} \lesssim \|\boldsymbol{\Phi}\|_{B^{1/2+\epsilon}_{q,q}} \lesssim \|\boldsymbol{\Phi}\|_{B^{0}_{q,q}} \sim \|\boldsymbol{\Phi}\|_{L^q} < \infty,$$

since $\boldsymbol{\Phi}$ is supported in the low frequency part in the Fourier space, and $\boldsymbol{\Phi} \in L^r(\mathbf{R}^2)$ for $2 < r < \infty$. This completes the proof of (4.10). □

参考文献

[1] L. Bergé, A. de Bouard, J. C. Saut, Blowing up time-dependent solutions of the planar, Chern-Simons gauged nonlinear Schrödinger equation, Nonlinearity **8** (1995), 235–253.

[2] T. Cazenave, "Semilinear Schrödinger equations," Courant Lecture Notes in Mathematics **10**, American Mathematical Society (2003).

[3] D. Chae, On the well-posedness of the Euler equations in the Triebel-Lizorkin spaces, Comm. Pure Appl. Math. **55** (2002), 654–678.

[4] N.-H. Chang, J. Shatah, K. Uhlenbeck, Schödinger maps, Comm. Pure Appl. Math. **53** (2000), 590–602.

[5] W. Ding, Y. Wang, Local Schrödinger flow into Kähler manifolds, Sci. China Ser. A **44** (2001), 1446–1464.

[6] L. D. Faddeev, L. A. Takhtajan, "Hamiltonian methods in the theory of solitons," Translated from the Russian by A. G. Reyman, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin (1987).

[7] M. G. Grillakis, V. Stefanopoulos, Lagrangian formulation, energy estimates, and the Schrödinger map problem, Comm. Partial Differential Equations **27** (2002), 1845–1877.

[8] J. Kato, Existence and uniqueness of the solution to the modified Schrödinger map, Math. Res. Lett. **12** (2005), 171–186.

[9] J. Kato, H. Koch, Uniqueness of the modified Schrödinger map in $H^{3/4+\varepsilon}(\mathbf{R}^2)$, Preprint.

[10] C. E. Kenig, K. D. Koenig, On the local well-posedness of the Benjamin-Ono and modified Benjamin-Ono equations, Math. Res. Lett. **10** (2003), 879–895.

[11] C. E. Kenig, A. Nahmod, The Cauchy problem for the hyperbolic-elliptic Ishimori system and Schrödinger maps, Nonlinearity **18** (2005), 1987–2009.

[12] H. Koch, N. Tzvetkov, On the local well-posedness of the Benjamin-Ono equation in $H^s(\mathbb{R})$, Int. Math. Res. Not. **26** (2003), 1449–1464.

[13] A. Nahmod, A. Stefanov, K. Uhlenbeck, On Schrödinger maps, Comm. Pure Appl. Math. **56** (2003), 114–151.

[14] A. Nahmod, A. Stefanov, K. Uhlenbeck, Erratum: On Schrödinger maps [Comm. Pure Appl. Math. **56** (2003), 114–151], Comm. Pure Appl. Math. **57** (2004), 833–839.

[15] T. Ogawa, A proof of Trudinger's inequality and its application to nonlinear Schrödinger equations, Nonlinear Anal. **14** (1990), 765–769.

[16] P.-L. Sulem, C. Sulem, C. Bardos, On the continuous limit for a system of classical spins, Comm. Math. Phys. **107** (1986), 431–454.

[17] E. M. Stein, "Singular integrals and differentiability properties of functions," Princeton Mathematical Series **30**, Princeton University Press (1970).

[18] M. V. Vladimirov, On the solvability of a mixed problem for a nonlinear equation of Schrödinger type, Soviet Math. Dokl. **29** (1984), 281–284.